Finally, consider the possibility that $a = b$. Setting $b = a$ in the original equation gives us

$$(2a)! = 22(a+c)!$$
$$\frac{(2a)!}{(a+c)!} = 22$$
$$(a+c+1)\cdots(2a) = 22.$$

Each factor on the left side is at least 4, so there can be at most two such factors. But 22 cannot be written as the product of two consecutive integers. So there must be exactly one factor on the left-hand side : $a+c+1 = 2a = 22$. Thus, $a = b = 11$ and $c = 10$, which also satisfies the original equation.

3968. *Proposed by Michal Kremzer.*

Let $\{a\} = a - [a]$, where $[a]$ is the greatest integer function. Show that if a is real and $a(a - 2\{a\})$ is an integer, then a is an integer.

We received 17 correct solutions, all with the same approach. We present the solution of Kathleen E. Lewis.

Since $\{a\} = a - [a]$, we have $a - 2\{a\} = [a] - \{a\}$. Thus

$$a(a - 2\{a\}) = ([a] + \{a\})([a] - \{a\}) = [a]^2 - \{a\}^2.$$

Since $[a]^2$ is an integer, $a(a-2\{a\})$ can only be an integer if $\{a\}^2$ is also an integer. But $0 \leq \{a\} < 1$ implies $0 \leq \{a\}^2 < 1$. Therefore, if $a - 2\{a\}$ is an integer, then $\{a\}$ must be zero and thus a must be an integer.

3969. *Proposed by Marcel Chiriţă.*

Determine the functions $f : (\frac{8}{9}, \infty) \mapsto \mathbb{R}$ continuous at $x = 1$ such that

$$f(9x-8) - 2f(3x-2) + f(x) = 4x - 4 + \ln \frac{9x^2 - 8x}{(3x-2)^2} \qquad \text{for all } x \in \left(\frac{8}{9}, \infty\right).$$

We received four correct solutions and one incomplete submission. We present two solutions.

Solution 1, by Arkady Alt.

We have the following

$$\begin{aligned} &f(9x-8) - 2f(3x-2) + f(x) \\ &\qquad = 4x - 4 + \ln \frac{9x^2 - 8x}{(3x-2)^2} \\ &\qquad = (9x-8) - 2(3x-2) + x + \ln x + \ln(9x-8) - 2\ln(3x-2), \end{aligned}$$

so $g(9x-8) - 2g(3x-2) + g(x) = 0$, where $g(x) := f(x) - \ln x - x$. Obviously $g : \left(\frac{8}{9}, \infty\right) \longrightarrow \mathbb{R}$ is continuous at $x = 1$.

Let $y := 9x - 8$, then $3x - 2 = \frac{y+2}{3}$ and the original functional equation becomes

$$g(y) - 2g\left(\frac{y+2}{3}\right) + g\left(\frac{y+8}{9}\right) = 0. \tag{1}$$

Consider the sequence $(x_n)_{n\geq 0}$ defined recursively by $x_{n+1} = \dfrac{x_n + 2}{3}, n \geq 0$ with initial condition $x_0 := x$, where $x \in \left(\frac{8}{9}, \infty\right)$ and $x \neq 1$.Then $x_n \in \left(\frac{8}{9}, \infty\right), n \geq 0$ and by replacing y in (1) with x_n we obtain

$$g(x_n) - 2g\left(\frac{x_n+2}{3}\right) + g\left(\frac{x_n+8}{9}\right) = 0$$

i.e.

$$g(x_n) - 2g\left(x_{n+1}\right) + g\left(x_{n+2}\right) = 0, \qquad n \geq 0.$$

Since $g(x_n) - g\left(x_{n+1}\right) = g\left(x_{n+1}\right) - g\left(x_{n+2}\right)$ for any $n \geq 0$ then by induction

$$g(x_n) - g\left(x_{n+1}\right) = g(x_0) - g\left(x_1\right).$$

On the other hand, since

$$x_{n+1} = \frac{x_n+2}{3} \iff x_{n+1} - 1 = \frac{1}{3}\left(x_n - 1\right),\ n \geq 0,$$

then

$$x_n - 1 = \frac{1}{3^n}\left(x_0 - 1\right) \iff x_n = \frac{x-1}{3^n} + 1.$$

Therefore, (by continuity in $x = 1$) we have $\lim\limits_{n\to\infty} g(x_n) = g(\lim\limits_{n\to\infty} x_n) = g(1)$.

Hence,

$$g(x) - g\left(\frac{x+2}{3}\right) = g(x_0) - g\left(x_1\right) = \lim_{n\to\infty}\left(g(x_n) - g\left(x_{n+1}\right)\right) = g(1) - g(1) = 0.$$

Since $g(1) - g\left(\frac{1+2}{3}\right) = 0$, then for any $x \in \left(\frac{8}{9}, \infty\right)$ we have $g(x) - g\left(\dfrac{x+2}{3}\right) = 0$ and therefore,

$$g(x_n) - g\left(\frac{x_n+2}{3}\right) = 0 \iff g(x_n) = g\left(x_{n+1}\right), \qquad n \geq 0.$$

Since $g(x_n) = g(x_0) = g(x)$ and $\lim\limits_{n\to\infty} g(x_n) = g(1)$ we obtain $g(x) = g(1)$ for any $x \in \left(\frac{8}{9}, \infty\right)$. Therefore, $f(x) = \ln x + x + c$, where c is any real constant.

Solution 2, by Digby Smith.

Lemma. Let g be a function satisfying $g(0) = 0$ which is continuous at $t = 0$ such that for $t \in \left(-\frac{1}{9}, \infty\right)$, the following equation holds :

$$g(9t) - 2g(3t) + g(t) = 0.$$

Then $g(t) = 0$ for all $t \in \left(-\frac{1}{9}, \infty\right)$.

Proof. Let $s \in \left(-\frac{1}{9}, \infty\right)$. For all $j \in \mathbb{N}$ with $j \geq 2$, the following holds :

$$g\left(9 \cdot \frac{s}{3^j}\right) - 2g\left(3 \cdot \frac{s}{3^j}\right) + g\left(\frac{s}{3^j}\right) = 0.$$

It then follows for $n \in \mathbb{N}$ with $n \geq 2$ that

$$\begin{aligned}
\sum_{j=2}^{n} \left[g\left(9 \cdot \frac{s}{3^j}\right) - 2g\left(3 \cdot \frac{s}{3^j}\right) + g\left(\frac{s}{3^j}\right)\right] &= 0,\\
g(s) - g\left(\frac{s}{3}\right) - g\left(\frac{s}{3^{n-1}}\right) + g\left(\frac{s}{3^n}\right) &= 0,\\
g(s) - g\left(\frac{s}{3}\right) = g\left(\frac{s}{3^{n-1}}\right) - g\left(\frac{s}{3^n}\right).
\end{aligned}$$

Since g is continuous at $t = 0$, it then follows that

$$g(s) - g\left(\frac{s}{3}\right) = \lim_{n\to\infty} \left(g\left(\frac{s}{3^{n-1}}\right) - g\left(\frac{s}{3^n}\right)\right) = g(0) - g(0) = 0,$$

so $g(s) = g\left(\frac{s}{3}\right)$. It then follows for $m \in \mathbb{N}$ that

$$g(s) = g\left(\frac{s}{3}\right) = g\left(\frac{s}{3^2}\right) = \ldots = g\left(\frac{s}{3^m}\right).$$

Again since g is continuous at $t = 0$, it follows that

$$g(s) = \lim_{m\to\infty} g\left(\frac{s}{3^m}\right) = g(0) = 0.$$

That is, $g(t) = 0$ for all $t \in \left(-\frac{1}{9}, \infty\right)$. □

Let $f(x) = x + \ln(x) + k + g(x-1)$ with $k \in \mathbb{R}$ and $g(t)$ continuous at $t = 0$ such that $g(0) = 0$ satisfy the functional equation

$$f(9x-8) - 2f(3x-2) + f(x) = 4x - 4 + \ln \frac{9x^2 - 8x}{(3x-2)^2}.$$

Substituting, we get that f satisfies the equation if and only if

$$g(9(x-1)) - 2g(3(x-1)) + g(x-1) = 0$$

for all $x \in \left(\frac{8}{9}, \infty\right)$. Applying the Lemma, it then follows that $g(x-1) = 0$ for all $x \in \left(\frac{8}{9}, \infty\right)$, which gives $f(x) = x + \ln(x) + k$.

Editor's Comments. Bataille noticed that $\frac{17}{18} > \frac{8}{9}$ while $9 \cdot \frac{17}{18} - 8 = \frac{1}{2} < \frac{8}{9}$ so that $f\left(9 \cdot \frac{17}{18} - 8\right)$ is not defined! The intended version of the problem seems to be :

> Determine the functions $f : (0, \infty) \to \mathbb{R}$ continuous at $x = 1$ such that
>
> $$f(9x-8) - 2f(3x-2) + f(x) = 4x - 4 + \ln \frac{9x^2 - 8x}{(3x-2)^2} \qquad \text{for all } x \in \left(\frac{8}{9}, \infty\right).$$